

# SUBELLIPTICITY OF THE $\bar{\partial}$ -NEUMANN PROBLEM ON A WEAKLY $Q$ -PSEUDOCONVEX/CONCAVE DOMAIN

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ABSTRACT. For a domain  $D$  of  $\mathbb{C}^n$  which is weakly  $q$ -pseudoconvex or  $q$ -pseudoconcave we give a sufficient condition for subelliptic estimates for the  $\bar{\partial}$ -Neumann problem. The paper extends to domains which are not necessarily pseudoconvex, the results and the techniques of Catlin [3].

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## 1. INTRODUCTION

Let  $D$  be a bounded domain of  $\mathbb{C}^n$  with smooth boundary. For a form  $f$  of degree  $k$  which satisfies  $\bar{\partial}f = 0$ , to solve the  $\bar{\partial}$ -Neumann problem consists in finding a form of degree  $k - 1$  such that

$$(1.1) \quad \begin{cases} \bar{\partial}u = f, \\ f \text{ is orthogonal to } \text{Ker } \bar{\partial}. \end{cases}$$

The main interest relies in the regularity at the boundary for this problem, that is, in stating under which condition  $u$  inherits from  $f$  the smoothness at the boundary  $\partial D$  (it certainly does in the interior). Let  $\bar{\partial}^*$  be the formal adjoint of  $\bar{\partial}$  under the choice of a smoothly varying hermitian metric on  $\bar{D}$ . Related to (1.1) is the problem

$$(1.2) \quad \begin{cases} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = f \\ u \in D_{\bar{\partial}} \cap D_{\bar{\partial}^*} \\ \bar{\partial}u \in D_{\bar{\partial}^*}, \bar{\partial}^*u \in D_{\bar{\partial}}, \end{cases}$$

where  $D_{\bar{\partial}^*}$  and  $D_{\bar{\partial}}$  are the domains of  $\bar{\partial}^*$  and  $\bar{\partial}$  respectively. This is a non-elliptic boundary value problem; in fact, the Kohn Laplacian  $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$  itself is elliptic but the boundary conditions which are imposed by the membership to  $D_{\square}$  are not. If (1.1) has a solution for every  $f$ , then one defines the  $\bar{\partial}$ -Neumann operator  $N := \square^{-1}$ ; this commutes both to  $\bar{\partial}$  and  $\bar{\partial}^*$ . If we then turn back to (1.1) and define  $u := \bar{\partial}^*Nf$  we see that

$$\begin{aligned} \bar{\partial}u &= \bar{\partial}^*Nf \\ &= \square Nf = f. \end{aligned}$$

Also,  $\bar{\partial}^* u = \bar{\partial}^* \bar{\partial}^* Nf = 0$  and therefore  $u$  is orthogonal to  $\text{Ker } \bar{\partial}$ . One of the main methods used in investigating the regularity at the boundary of the solutions of (1.1) consists in certain a priori subelliptic estimates.

**Definition 1.1.** The  $\bar{\partial}$ -Neumann problem is said to satisfy a subelliptic estimate of order  $\epsilon > 0$  at  $z_o \in \bar{D}$  on  $k$  forms if there exist a positive constant  $c$  and a neighborhood  $V \ni z_o$  such that

$$(1.3) \quad \|u\|_\epsilon \leq c(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) \quad \text{for any } u \in C_c^\infty(\bar{D} \cap V)^k \cap D_{\bar{\partial}^*}.$$

By Garding's inequality, subelliptic estimates of order 1, that is, elliptic estimates hold in the interior of  $D$ . So our interest is confined to the boundary  $\partial D$ . When the domain  $D$  is pseudoconvex, a great deal of work has been done about subelliptic estimates. The most general results concerning this problem have been obtained by Kohn [11] and Catlin [3].

In [11], Kohn gave a sufficient condition for subellipticity over pseudoconvex domains with real analytic boundary by introducing a sequence of ideals of subelliptic multipliers.

In [3], Catlin proved, regardless whether  $\partial D$  is real analytic or not, that subelliptic estimates hold for  $k$  forms at  $z_o$  if and only if a certain number  $D_k(z_o)$  is finite. Note that the definition of  $D_k(z_o)$  in [3] is closely related to that of  $\Delta_k(z_o)$  due to D'Angelo. In particular, when  $k = 1$ , these numbers do coincide.

However, not much is known in the case when the domain is not necessarily pseudoconvex except from the results related to the celebrated  $Z(k)$  condition which characterizes the existence of subelliptic estimates for  $\epsilon = \frac{1}{2}$  according to Hörmander [13] and Folland-Kohn [5]. Some further results, mainly related to the case of forms of top degree  $n - 1$  are due to Ho [15].

We exploit here the full strength of Catlin's method to study subellipticity on domains which are not necessarily pseudoconvex. Let  $\partial D$  be defined by  $r = 0$  with  $r < 0$  on the side of  $D$  and let  $T^{\mathbb{C}}\partial D$  be the complex tangent bundle to  $\partial D$ . We use the following notations:  $L_{\partial D} = (r_{ij})|_{T^{\mathbb{C}}\partial D}$  is the Levi form of the boundary,  $s_{\partial D}^+$ ,  $s_{\partial D}^-$ ,  $s_{\partial D}^0$  are the numbers of eigenvalues of  $L_{\partial D}$  which are  $> 0$ ,  $< 0$ ,  $= 0$  respectively and finally  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1}$  are its ordered eigenvalues. We take a pair of indices  $1 \leq q \leq n - 1$  and  $0 \leq q_o \leq n - 1$  such that  $q \neq q_o$ . We assume that there is a bundle  $\mathcal{V}^{q_o} \subset T^{1,0}\partial D$  of rank  $q_o$  with smooth coefficients in  $V$ , say the bundle of the first  $q_o$  coordinate tangential vector fields  $L_1, \dots, L_{q_o}$ , such that

$$(1.4) \quad \sum_{j=1}^q \lambda_j - \sum_{j=1}^{q_o} r_{jj} \geq 0 \quad \text{on } \partial D.$$

**Definition 1.2.** (i) If  $q > q_o$  we say that  $D$  is  $q$ -pseudoconvex.  
(ii) If  $q < q_o$  we say that  $D$  is  $q$ -pseudoconcave.

*Remark 1.3.* The notion of  $q$ -pseudoconvexity was used in [17] to prove the existence of  $C^\infty(\bar{D})$  solutions to the equation  $\bar{\partial}u = f$ . Though the notion of  $q$ -pseudoconcavity is

formally symmetric to  $q$ -pseudoconvexity, it is useless in the existence problem. The reason is intrinsic. Existence is a “global” problem but bounded domains are never globally  $q$ -pseudoconcave. Owing to the local nature of subelliptic estimates and the related hypoellipticity of  $\bar{\partial}$ , this is the first occurrence where  $q$ -pseudoconcavity comes successfully into play.

*Remark 1.4.* Assume that (1.4) holds. Then, if  $q > q_o$ , we must have  $\lambda_q \geq 0$ . Thus (1.4) still holds with the first sum  $\sum_{j=1}^q$  replaced by  $\sum_{j=1}^k$  for any  $k \geq q$ . Similarly, if  $q < q_o$ , then  $\lambda_{q+1} \leq 0$ . Thus (1.4) holds with  $\sum_{j=1}^q$  replaced by  $\sum_{j=1}^k$  for any  $k \leq q$ .

*Remark 1.5.* When we have strict inequality “ $<$ ” in (1.4), it means that we have in fact  $\lambda_q > 0$  and  $\lambda_{q+1} < 0$  in the two cases respective  $q > q_o$  and  $q < q_o$ . It follows

$$(1.5) \quad q > n - 1 - s^+ \quad (\text{resp. } q < s^-).$$

We refer to these two situations as “strong”  $q$ -pseudoconvexity (resp. -pseudoconcavity). Note that this is the same as to saying, in the terminology of Folland-Kohn, that  $D$  satisfies  $Z(k)$  for any  $k \geq q$  (resp.  $k \leq q$ ).

We write  $k$ -forms as  $u = (u_J)_J$  where  $J = j_1 < j_2 < \dots < j_k$  are ordered multiindices. When the multiindices are not ordered, the coefficients are assumed to be alternant. Thus, if  $J$  decomposes as  $J = jK$ , then  $u_{jK} = \text{sign}\left(\begin{smallmatrix} J \\ jK \end{smallmatrix}\right) u_J$ . We define the  $\delta$ -strip of  $D$  along the boundary by  $S_\delta = \{z \in D : r(z) > -\delta\}$ . The main result in this paper is the following.

**Theorem 1.6.** *Let (1.4) be satisfied in a neighborhood of  $z_o$ , let  $k \geq q$  (resp.  $k \leq q$ ) for  $q > q_o$  (resp.  $q < q_o$ ) and suppose that for small  $\delta$  there exists a weight  $\varphi = \varphi^\delta$  in  $C^2(\bar{D} \cap V)$  such that*

$$(1.6) \quad \left\{ \begin{array}{l} \varphi \leq 1, \\ \sum_{|K|=k-1} ' \sum_{i,j=1}^n \varphi_{ij}(z) u_{iK} \bar{u}_{jK} - \sum_{|J|=k} ' \sum_{j=1}^{q_o} \varphi_{jj} |u_J|^2 \\ \geq c \sum_{j=1}^{q_o} |L_j(\varphi)|^2 |u|^2, \text{ for any } z \in \bar{D} \cap V \end{array} \right.$$

and

$$(1.7) \quad \left\{ \begin{array}{l} |\varphi| \leq 1, \\ \sum_{|K|=k-1} ' \sum_{i,j=1}^n \varphi_{ij}(z) u_{iK} \bar{u}_{jK} - \sum_{|J|=k} ' \sum_{j=1}^{q_o} \varphi_{jj} |u_J|^2 \\ \geq c\delta^{-2\epsilon} |u|^2 \text{ for any } z \in \bar{S}_\delta \cap V, \end{array} \right.$$

where the constant  $c > 0$  does not depend on  $\delta$  or  $u$ .

Then,  $\epsilon$ -subelliptic estimates at  $z_o$  hold for forms of degree  $k$ .

It is not restrictive to assume, as we will do all throughout the paper, that  $L_j(z_o) = \partial_{z_j}$  for any  $j$ . For every  $q$ -pseudoconvex/concave domain there is a small perturbation for which subelliptic estimates hold.

**Theorem 1.7.** *Let  $D$  be  $q$ -pseudoconvex (resp.  $q$ -pseudoconcave); thus it is defined by  $r < 0$  for  $r := 2\operatorname{Re} z_n + h(z_1, \dots, z_{n-1}, y_n)$  satisfying (1.4) for  $q > q_o$  (resp.  $q < q_o$ ). Let  $\tilde{r} := r + \sum_{j=q}^{n-1} h_j(z_j)$  (resp.  $\tilde{r} = r - \sum_{j=1}^{q+1} h_j(z_j)$ ) where the  $h_j$ 's are real positive subharmonic, non harmonic, functions of vanishing order  $2m_j$  that, by reordering, we may assume to be decreasing  $\dots m_j \geq m_{j+1} \dots$  (resp. increasing  $\dots m_j \leq m_{j+1} \dots$ ) and let  $\tilde{D}$  be defined by  $\tilde{r} < 0$ .*

*Then subelliptic estimates hold for  $\tilde{D}$  in degree  $k \geq q$  (resp.  $k \leq q$ ) of any order  $< \epsilon_k$  for  $\epsilon_k := \frac{1}{2m_k}$  (resp.  $\epsilon_k := \frac{1}{2m_{k+1}}$ ). In both cases, when  $\epsilon_k = \frac{1}{2}$ , we have in fact estimates including for order  $\frac{1}{2}$ .*

When  $\epsilon_k = \frac{1}{2}$  it means that  $Z(k)$  is satisfied; thus we regain the result by Hörmander and Folland-Kohn. We will refer to functions such as the above  $h_j$ 's as subharmonic functions satisfying  $h_j \cong |z_j|^{2m_j}$ .

*Example 1.8.* Let  $D$  be defined by

$$2\operatorname{Re} z_n - \sum_{j=1}^{q_o} |z_j|^{2m_j} + \sum_{j=q_o+1}^{n-1} |z_j|^{2m_j} < 0,$$

where the two groups of indices  $\{m_1, \dots, m_{q_o}\}$  and  $\{m_{q_o+1}, \dots, m_{n-1}\}$  have increasing and decreasing order respectively. Then subelliptic estimates hold in degree  $k \neq q_o$  of any order smaller than  $\epsilon_k$  defined in Theorem 1.7.

**Corollary 1.9.** *Let  $D$  be a domain in  $\mathbb{C}^n$  defined by*

$$2\operatorname{Re} z_n + g + |z_{n-1}|^{2m} < 0$$

*where  $g$  is a real  $C^\infty$  function such that  $g_{n-1, n-1} = o(|z_{n-1}|^{2(m-1)})$ . Then subelliptic estimates of order  $\epsilon < \frac{1}{2m}$  hold at  $z_o = 0$  for any  $(n-1)$ -form.*

*Proof.* Put  $r := 2\operatorname{Re} z_n + g + \frac{1}{2}|z_{n-1}|^{2m}$ ; we claim that  $r$  satisfies (1.4) for  $q_o = n-2$  and  $q = n-1$ . In fact for a tangential  $(n-1)$ -form the only coefficient which does not vanish at  $\partial D$  is  $u_J = u_{1, \dots, n-1}$ . Thus

$$\sum'_{|K|=k-1} \sum_{i,j=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} - \sum'_{|J|=k} \sum_{j=1}^{n-2} r_{jj} |u_J|^2 = (|z_{n-1}|^{2(m-1)} + o(|z_{n-1}|^{2(m-1)})) |u_{1, \dots, n-1}|^2$$

which is  $\geq 0$ . We are thus in position to apply Theorem 1.7.

□

*Example 1.10.* Let  $D$  be defined by

$$2\operatorname{Re} z_3 - |z_1^2 + z_2^3|^2 \pm |z_1|^{2m} + |z_2|^4 < 0 \quad \text{or} \quad 2\operatorname{Re} z_3 - |z_1^2 z_2^3|^2 \pm |z_1|^{2m} + |z_2|^4 < 0;$$

then subelliptic estimates hold at  $z_o = 0$  on 2-forms for any order  $\epsilon < \frac{1}{4}$ .

Remark : Corollary 1.9 is more general than Corollary 3.4 in [16] where  $g$  cannot depend on  $z_{n-1}$  and  $y_n$ .

We decompose the coordinates as  $z = (z', z'', z_n) \in \mathbb{C}^{q_o} \times \mathbb{C}^{n-q_o-1} \times \mathbb{C}$ . The conclusion contained in Theorem 1.7 is sharp.

**Theorem 1.11.** (i) Let  $r = 2\operatorname{Re} z_n - Q(z')$  for  $Q \geq 0$  and set  $\tilde{r} = r + \sum_{j=q_o+1}^{n-1} h_j(z_j)$  where the  $h_j$ 's are subharmonic and satisfy  $h_j \cong |z_j|^{2m_j}$  with  $m_j \geq m_{j+1} \geq \dots$  (decreasing) and with  $Q = O(|z'|^{2m_{q_o+1}})$ . If  $\epsilon$ -subelliptic estimates at  $z_o = 0$  hold in degree  $k \geq q_o + 1$ , then we must have  $\epsilon \leq \frac{1}{2m_k}$ .

(ii) Let  $r = 2\operatorname{Re} z_n + Q(z'')$  for  $Q \geq 0$  and set  $\tilde{r} = r + \sum_{j=1}^{q_o} h_j(z_j)$  with  $h_j$  subharmonic satisfying  $h_j \cong |z_j|^{2m_j}$  with  $m_j \leq m_{j+1} \leq \dots$  (increasing). We also assume  $m_1 \geq \frac{m_{q_o-1}}{2} + \frac{1}{4}$  and  $Q = O(|z''|^{2m_{q_o-1}})$ . If  $\epsilon$ -subelliptic estimates hold at  $z_o = 0$  in degree  $k \leq q_o - 1$ , then  $\epsilon \leq \frac{1}{2m_k}$ .

Necessary conditions for subellipticity in degree  $k = n - 1$  are also stated in [15]; however, the  $\bar{\partial}$ -Neumann conditions seem not to be respected in the proof therein.

The paper is structured as follows. In Section 2 we introduce the geometric concept of  $q$ -pseudoconvexity and  $q$ -pseudoconcavity. In Section 3 we derive some basic inequalities which are useful for the proof of Theorem 1.2. Sections 4, 5 and 6 are devoted to the proof of Theorem 1.6, Theorem 1.7 and Theorem 1.11 respectively.

## 2. $Q$ -PSEUDOCONVEX/PSEUDOCONCAVE DOMAINS

Let  $D$  be a bounded domain in  $\mathbb{C}^n$  with smooth boundary  $\partial D$  defined by  $r = 0$  with  $\partial r \neq 0$ . For a given boundary point  $z_o \in \partial D$ , we consider a complex frame adapted to  $\partial D$ , that is, an orthonormal basis  $\omega_1, \dots, \omega_n = \partial r$  of  $(1, 0)$  forms with  $C^\infty$  coefficients in a neighborhood of  $z_o$ . We denote by  $(r_{jk}(z))_{j,k=1}^n$  the matrix of the Levi form  $\partial \bar{\partial} r(z)$  with respect to the basis  $\omega^1, \dots, \omega^n$ . Let  $\lambda_1(z) \leq \dots \leq \lambda_{n-1}$  be the eigenvalues of  $(r_{jk}(z))_{j,k=1}^{n-1}$  and denote  $s_{\partial D}^+, s_{\partial D}^-, s_{\partial D}^0$  their number according to the different sign. Let  $q$  and  $q_o$  be a pair of indices for which (1.4) is fulfilled in a suitable choice of the frame; remember that we have defined  $D$  to be  $q$ -pseudoconvex or  $q$ -pseudoconcave according to  $q > q_o$  or  $q < q_o$ ; for the case  $q > q_o$  this definition follows [17]. The pseudoconvexity/concavity is said to be strong when (1.4) holds as strict inequality.

As it has already been noticed, (1.4) for  $q > q_o$  implies  $\lambda_q \geq 0$ ; hence (1.4) is still true if we replace the first sum  $\sum_{j=1}^q \cdot$  by  $\sum_{j=1}^k \cdot$  for any  $k$  such that  $q \leq k \leq n-1$ . Similarly, if it holds for  $q < q_o$ , then  $\lambda_{q+1} \leq 0$  and hence it also holds with  $q$  replaced by  $k \leq q$  in the first sum.

*Example 2.1.* It is readily seen that for  $q_o = s^- + s^0$  and for any  $q > q_o$  (resp.  $q_o = s^-$  and any  $q < q_o$ ), (1.4) is satisfied in a suitable local boundary frame. Thus any index  $q \notin [s^-, s^- + s^0]$  satisfies (1.4) for either choice of  $q_o$ . The interesting point in (1.4) is to be capable to get (1.4) for indices  $q \notin [s^-, s^- + s^0]$ .

*Example 2.2.* Let  $s^-(z)$  be constant for  $z \in \partial D$  close to  $z_o$ ; then (1.4) holds for  $q_o = s^-$  and  $q = s^- + 1$ . In fact, we have  $\lambda_{s^-} < 0 \leq \lambda_{s^-+1}$ , and therefore the negative eigenvectors span a bundle  $\mathcal{V}^{q_o}$  for  $q_o = s^-$  that, identified with the span of the first  $q_o$  coordinate vector fields, yields  $\sum_{j=1}^{q_o+1} \lambda_j(z) \geq \sum_{j=1}^{q_o} r_{jj}(z)$ . Note that a pseudoconvex domain is characterized by  $s^-(z) \equiv 0$ , thus, it is 1-pseudoconvex in our terminology.

In the same way, if  $s^+(z)$  is constant at  $z_o$ , then  $\lambda_{s^-+s^0} \leq 0 < \lambda_{s^-+s^0+1}$ . Then, the eigenspace of the eigenvectors  $\leq 0$  is a bundle which, identified to that of the first  $q_o = s^- + s^0$  coordinate vector fields yields (1.4) for  $q = q_o - 1$ . In particular a pseudoconcave domain, that is a domain which satisfies  $s^+ \equiv 0$ , is  $(n-2)$ -pseudoconcave in our terminology.

The following lemma plays an essential role in the following.

**Lemma 2.3.** *Let  $D$  be a smoothly bounded domain. Then  $D$  is  $q$ -pseudoconvex (resp.  $q$ -pseudoconcave) if and only if*

$$(2.1) \quad \sum_{|K|=k-1} ' \sum_{i,j=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} - \sum_{|J|=k} ' \sum_{i=1}^{q_o} r_{jj} |u_J|^2 \geq 0 \quad \text{on } \partial D,$$

for any  $u$  of degree  $k \geq q > q_o$  (resp.  $k \leq q < q_o$ ) satisfying  $u_{nK}|_{\partial D} = 0$  for any  $K$ .

The proof is the same as in [17].

For convenient writing, we shall use the notation  $A \lesssim B$  to mean  $A \leq cB$  for some constant  $c$ , which is independent of relevant parameters. And  $A \cong B$  if  $A \lesssim B$  and  $B \lesssim A$ .

### 3. THE BASIC ESTIMATES ON Q-PSEUDOCONVEXITY/CONCAVITY

In this section we prepare some inequalities which are needed for the subelliptic estimates of our Theorem 1.6. The key technical tool of our discussion are the so call Hormander-Kohn-Morrey estimates contained in the following proposition. Let  $D$  be a domain with smooth boundary defined by  $r = 0$  in a neighborhood of  $z_o$ . Let  $\omega_1, \dots, \omega_n = \partial r$  be an orthonormal basis of  $(1, 0)$  forms and  $L_1, \dots, L_n$  the dual basis of  $(1, 0)$  vector fields.

For  $0 \leq k \leq n$ , we write a general  $k$ -form  $u$  as

$$u = \sum_{|J|=k} ' u_J \bar{\omega}_J,$$

where  $\sum'$  denotes summation restricted to ordered multiindices  $J = \{j_1, \dots, j_k\}$  and where  $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge \dots \wedge \bar{\omega}_{j_k}$ . When the multiindex is no more ordered, it is understood that the coefficient  $u_J$  is an antisymmetric function of  $J$ ; in particular, if  $J$  decomposes into  $jK$ , then  $u_{jK} = \text{sign}\binom{J}{jK} u_J$ . We define  $\langle u, u \rangle$  by  $\langle u, u \rangle = |u|^2 = \sum'_{|J|=k} |u_J|^2$ ; this definition is independent of the choice of orthonormal basis  $\omega_1, \dots, \omega_n$ . The coefficients of our forms are taken in various spaces  $\Lambda$  such as  $C^\infty(\bar{D})$ ,  $C^\infty(D)$ ,  $C_c^\infty(\bar{D})$ ,  $L^2(D)$  and the corresponding spaces of  $k$ -forms are denoted by  $\Lambda^k$ . Though our a priori estimates are proved over smooth forms, they are stated in Hilbert norms. Thus, let  $\|u\|$  be the  $H^0 = L^2$  norm and, for a real function  $\varphi$ , let the weighted  $L^2$ -norm be defined by

$$\|u\|_{H_\varphi^0}^2 := \sum'_{|J|=k} \int_D e^{-\varphi} |u_J|^2 dv$$

where  $dv$  is the element of volume in  $\mathbb{C}^n$ . We begin by noticing that  $\bar{\partial}$  is closed, densely defined. Also, its domain  $D_{\bar{\partial}}$  certainly contains smooth forms and its action is expressed by

$$(3.1) \quad \bar{\partial}u = \sum'_{|K|=k-1} \sum_{\substack{ij=1, \dots, n \\ i < j}} (\bar{L}_i u_{jK} - \bar{L}_j u_{iK}) \bar{D}_i \wedge \bar{\omega}_j \wedge \bar{\omega}_K + \dots,$$

where dots denote terms in which no differentiation of  $u$  occurs.

Let  $\bar{\partial}^*$  be the adjoint of  $\bar{\partial}$ . The operator  $\bar{\partial}^*$  is still closed, densely defined but it is no more true that smooth forms belong to  $D_{\bar{\partial}^*}$ . For this, they must satisfy certain boundary conditions. Namely, integration by parts shows that a form  $u$  of degree  $k$  cannot belong to  $D_{\bar{\partial}^*}$  unless

$$\sum_{j=1}^n \int_{\partial D} e^{-\varphi} L_j(r) u_{jK} \psi_K ds = 0 \quad \text{for any } K \text{ and any } \psi_K \text{ of degree } k-1.$$

This means that  $\sum_{j=1}^n L_j(r) u_{jK}|_{\partial D} \equiv 0$  for any  $K$ . (Here  $ds$  is the element of hypersurface in  $\partial D$ .) Since we have chosen our basis with the property  $L_j(r)|_{\partial D} = \kappa_{jn}$  (the Kronecker's symbol), we then conclude

$$(3.2) \quad u \text{ belongs to } D_{\bar{\partial}^*} \text{ iff } u_J|_{\partial D} = 0 \text{ whenever } n \in J.$$

We call *tangential* a form which belongs to  $D_{\bar{\partial}^*}$ . Let  $\mathcal{L}_j^\varphi$  be the formal  $H_\varphi^0$ -adjoint of  $-L_j$ ; over a tangential form the action of the Hilbert adjoint of  $\bar{\partial}$ , coincides with that of its “formal adjoint” and is therefore expressed by a “divergence operator”:

$$(3.3) \quad \bar{\partial}_\varphi^* u = - \sum'_{|K|=k-1} \sum_j \mathcal{L}_j^\varphi(u_{jK}) \bar{\omega}_K + \dots \quad \text{for any } u \in D_{\bar{\partial}^*},$$

where dots denote an error term in which  $u$  is not differentiated and  $\varphi$  does not occur. By developing the equalities (3.1) and (3.3) by means of integration by parts, we get the proof of the following crucial result.

**Proposition 3.1.** *Let  $D$  be a smoothly bounded domain and fix arbitrarily an index  $q_0$  with  $0 \leq q_0 \leq n-1$ . Then for a suitable  $C > 0$  and any  $u \in C^\infty(\bar{D})^k \cap D_{\bar{\partial}^*}$ , we have*

$$(3.4) \quad \|\bar{\partial}u\|_{H_\varphi^0}^2 + \|\bar{\partial}^*u\|_{H_\varphi^0}^2 + C\|u\|_{H_\varphi^0}^2 \geq$$

$$(3.5) \quad + \sum_{|K|=k-1} ' \sum_{i,j=1}^n \int_D e^{-\varphi} \varphi_{ij} u_{iK} \bar{u}_{jK} dv - \sum_{|J|=k} ' \sum_{j=1}^{q_0} \int_D e^{-\varphi} \varphi_{jj} |u_J|^2 dv$$

$$(3.6) \quad + \sum_{|K|=k-1} ' \sum_{i,j=1}^{n-1} \int_{\partial D} e^{-\varphi} r_{ij} u_{iK} \bar{u}_{jK} ds - \sum_{|J|=q} ' \sum_{j=1}^{q_0} \int_{\partial D} e^{-\varphi} r_{jj} |u_J|^2 ds$$

$$(3.7) \quad + (1-\alpha) \left( \sum_{j=1}^{q_0} \|\mathcal{L}_j^\varphi u\|_{H_\varphi^0}^2 + \sum_{j=q_0+1}^n \|\bar{L}_j u\|_{H_\varphi^0}^2 \right).$$

We refer for instance to [18] for the proof of Proposition 3.1. We note that there is no relation between  $k$  and  $q_0$  in above inequality and that  $C$  and  $\alpha$  are independent of  $\varphi$  (and  $u$ ). By choosing  $\varphi$  so that  $e^{-\varphi}$  is bounded, we may remove the weight functions in (3.4) to get some inequalities that are useful for the proof of Theorem 1.6. We write  $Q(u, u) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$ .

**Theorem 3.2.** *Assume that the hypotheses of Theorem 1.6 be fulfilled. Then, for a suitable neighborhood  $V$  of  $z_o$  and for  $\delta$  small, we have*

$$(3.8) \quad \|u\|^2 + \delta^{-2\epsilon} \int_{S_\delta} |u|^2 dv + \sum_{j=1}^{q_0} \|L_j u\|^2 + \sum_{j=q_0+1}^n \|\bar{L}_j u\|^2 \lesssim Q(u, u)$$

for any  $u \in C_c^\infty(\bar{D} \cap V)^k \cap D_{\bar{\partial}^*}$  with  $k \geq q$  (resp.  $k \leq q$ ) when  $q > q_0$  (resp.  $q < q_0$ ).

*Proof.* We use twice Proposition 3.1 and in both cases, owing to the assumption of  $q$ -pseudoconvexity (resp.  $q$ -pseudoconcavity) we have the crucial fact that the boundary integrals are  $\geq 0$  for any  $k \geq q > q_0$  (resp.  $k \leq q < q_0$ ). We first use Proposition 3.1 under the choice  $\varphi \equiv 0$  and get

$$(3.9) \quad Q(u, u) + C\|u\|^2 \gtrsim \sum_{j=1}^{q_0} \|L_j u\|^2 + \sum_{j=q_0+1}^n \|\bar{L}_j u\|^2 \quad u \in C^\infty(\bar{D} \cap U)^k \cap D_{\bar{\partial}^*}.$$



We use again Proposition 3.1, this time for  $\varphi = \chi(\varphi^\delta)$ . In this case, the second line of (3.4) splits into two terms

$$(3.10) \quad \int_D e^{-\chi(\varphi^\delta)} \dot{\chi} \left( \sum'_{|K|=k-1} \sum_{i,j=1}^n \varphi_{ij}^\delta u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_0} \varphi_{jj}^\delta |u|^2 \right) dv \\ + \int_D e^{-\chi(\varphi^\delta)} \ddot{\chi} \left( \sum'_{|K|=k-1} \left| \sum_{j=1}^n \varphi_j^\delta u_{jK} \right|^2 - \sum_{j=1}^{q_0} |\varphi_j^\delta|^2 |u|^2 \right) dv.$$

We also have

$$(3.11) \quad \|\bar{\partial}_{\chi(\varphi^\delta)}^* u\|_{H^0_{\chi(\varphi^\delta)}}^2 \leq 2 \|\bar{\partial}^* u\|_{H^0_{\chi(\varphi^\delta)}}^2 + 2 \sum'_{|K|=k-1} \|\dot{\chi}^2 \sum_{j=1}^n \varphi_j^\delta u_{jK}\|_{H^0_{\chi(\varphi^\delta)}}^2.$$

Remark that  $|\sum_{j=1}^{q_0} (\chi(\varphi^\delta))_j u|^2 = |\dot{\chi}|^2 |\sum_{j=1}^{q_0} \varphi_j^\delta|^2 |u|^2$ . Thus we get from (3.4), under the choice of the weight  $\chi(\varphi^\delta)$ , and taking into account (3.10) and (3.11):

$$(3.12) \quad \|\bar{\partial} u\|_{H^0_{\chi(\varphi^\delta)}}^2 + 2 \|\bar{\partial}_{\chi(\varphi^\delta)}^* u\|_{H^0_{\chi(\varphi^\delta)}}^2 + 2C \|u\|_{H^0_{\chi(\varphi^\delta)}}^2 \\ \geq \int_D \dot{\chi} e^{-\chi(\varphi^\delta)} \left( \sum'_{|K|=k-1} \sum_{i,j=1}^n \varphi_{ij}^\delta u_{iK} \bar{u}_{jK} dv - \sum'_{|J|=q_0} \sum_{j=1}^{q_0} \varphi_{jj}^\delta |u_J|^2 \right) dv \\ + \int_D (\ddot{\chi} - 2\dot{\chi}^2) e^{-\chi(\varphi^\delta)} \sum'_{|K|=k-1} \left| \sum_{j=1}^n \varphi_j^\delta u_{jK} \right|^2 dv - \int_D \ddot{\chi} e^{-\chi(\varphi^\delta)} \sum_{j=1}^{q_0} |\varphi_j^\delta|^2 |u|^2 dv.$$

We now specify our choice of  $\chi$ . First, we want  $\ddot{\chi} \geq 2\dot{\chi}^2$  so that the first sum in the third line can be disregarded. Keeping this condition, we need an opposite estimate which assures that the absolute value of the last negative term in the third line of (3.12) is controlled by one half of the second line. If  $c$  is the constant of (1.6), the above condition is fulfilled as soon as  $\frac{2\ddot{\chi}}{\dot{\chi}} \leq c$ . If we then set  $\chi := \frac{1}{2} e^{\frac{c}{2}(t-1)}$  then both requests are satisfied; (we also notice that  $\dot{\chi}^2 \ll \dot{\chi}$  because  $c \ll 1$ ). Thus our inequality continues as

$$(3.13) \quad \geq \frac{1}{2} \int_D \dot{\chi} e^{-\chi(\varphi^\delta)} \left( \sum'_{|K|=k-1} \sum_{i,j=1}^n \varphi_{ij}^\delta u_{iK} \bar{u}_{jK} dv - \sum'_{|J|=q_0} \sum_j^q \varphi_{jj}^\delta |u_J|^2 \right) dv \\ \geq \frac{1}{2} \int_{S_\delta} \dot{\chi} e^{-\chi(\varphi^\delta)} \left( \sum'_{|K|=k-1} \sum_{i,j=1}^n \varphi_{ij}^\delta u_{iK} \bar{u}_{jK} - \sum'_{|J|=k} \sum_j^q \varphi_{jj}^\delta |u_J|^2 \right) dv \\ \geq \delta^{-2\epsilon} \int_{S_\delta} \frac{c}{2} \dot{\chi} e^{-\chi(\varphi^\delta)} |u|^2 dv.$$

Here we are using the two main assumptions for our weights  $\varphi^\delta$ , that is, (1.6)(with the right side replaced by 0) to get the second inequality and (1.7) as for the third. Thus the first line of (3.10) is bigger or equal to the last of (3.13). We want to remove the weight from the resulting inequality. The first term can be handled owing to  $e^{-\chi(\varphi^\delta)} \leq 1$  on  $\bar{D} \cap V$  and the second owing to  $\dot{\chi}e^{-\chi(\varphi^\delta)} \geq c \geq 0$  on  $S_\delta \cap V$  which follows in turn from  $|\varphi^\delta| < 1$ . We end up with the unweighted estimate

$$(3.14) \quad \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + C\|u\|^2 \gtrsim \delta^{-2\epsilon} \int_{S_\delta} |u|^2 dv.$$

Now, for fixed  $\delta_o$  and for  $V$  contained in the  $\delta_o$ -ball centered at  $z_o = 0$ , the term  $C\|u\|^2$  in the left of (3.14) can be absorbed in the right. Thus we end up with the estimate

$$(3.15) \quad \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 \gtrsim \delta^{-2\epsilon} \int_{S_\delta} |u|^2 dv + \|u\|^2$$

for any  $u \in C_c^\infty(\bar{D} \cap V)^k \cap D_{\bar{\partial}^*}$  and  $\delta \leq \delta_o$ .

Combining (3.9) and (3.15), we get (3.8) which concludes the proof of the theorem.  $\square$

#### 4. PROOF OF THEOREM 1.6

Let  $V$  be a neighborhood of a given point  $z_o \in \partial D$ , let  $(t, r)$  be smooth coordinates in  $V$  with  $t = (t_1, \dots, t_{2n-1})$  and let  $\tau$  be dual coordinates to  $t$ . For a function  $u$  supported in  $V$ , one defines the tangential Fourier transform by

$$\hat{u}(\tau, r) = \int_{\mathbb{R}^{2n-1}} e^{-it\tau} u(t, r) dt,$$

and the tangential  $H^s$ -Sobolev norm by

$$|||u|||_s^2 = \|\Lambda^s u\|^2 = \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} (1 + |\tau|^2)^s |\hat{u}(\tau, r)|^2 d\tau dr,$$

where  $\Lambda^s$  is the tangential Bessel potential of order  $s$ . We note that when  $s = 0$  then  $|||u|||_0 = \|u\|$  is the usual  $L^2$ -norm. We refer to [5] for further details.

We remark that if  $D_i$  is  $\frac{\partial}{\partial t_j}$  or  $\frac{\partial}{\partial r}$  then

$$\begin{aligned} |||u|||_s^2 &= \sum_i |||D_i u|||_s^2 \\ &\cong |||u|||_{s+1}^2 + |||D_r u|||_s^2. \end{aligned}$$

The next result contains the key estimate in the proof of Theorem 1.6.

**Lemma 4.1.** *Let  $U$  be a special boundary chart for  $D$ . Then for all  $z_o \in \partial D \cap U$  there exists a neighborhood  $V \subseteq U$  of  $z_o$  such that*

$$|||u|||_\epsilon^2 \lesssim \sum_{j \leq q_0} |||L_j u|||_{\epsilon-1}^2 + \sum_{j \geq q_0+1} |||\bar{L}_j u|||_{\epsilon-1}^2 + ||u_b||_{\epsilon-\frac{1}{2}}^2, \quad u \in C^\infty(V \cap \bar{D})^k \cap D_{\bar{\partial}*}$$

where  $u_b := u|_{\partial D}$  and  $\epsilon \leq \frac{1}{2}$

The above lemma is a variant of Theorem (2.4.5) of [5] to which we refer for the proof. Notice that on one hand our statement is more general because we choose any  $\epsilon \leq \frac{1}{2}$  instead of  $\epsilon = \frac{1}{2}$ . On the other, we specialize the choice of a general elliptic system to the case of  $\{L_j\}_{j \leq q_0} \cup \{\bar{L}_j\}_{q_0+1 \leq j \leq n}$ .

For the proof of Theorem 1.6, we use a method derived from [3]. Let  $p_k(t), k = 0, 1, \dots$  be a sequence of functions with  $\sum_{k=0}^\infty p_k^2(t) = 1$ ,  $p_k(t) \equiv 0$  if  $t \notin (2^{k-1}, 2^{k+1})$  with  $k \geq 1$  and  $p_0(t) \equiv 0, t \geq 2$ . We can also choose  $p_k$  so that

$$|p'_k(t)| \leq C2^{-k}.$$

Let  $P_k$  denote the operator defined by

$$(\widehat{P_k u})(\tau, r) = p_k(|\tau|)\hat{u}(\tau, r)$$

where  $\hat{u}$  is the tangential Fourier transform. Let  $\mathbb{R}_-^{2n} := \{z : r(z) < 0\}$  and denote by  $\mathcal{S}(\mathbb{R}_-^{2n})$  the Schwartz space of  $C^\infty(\mathbb{R}_-^{2n})$ -functions which are rapidly decreasing at  $\infty$ .

**Lemma 4.2.** *For  $f, u \in \mathcal{S}(\mathbb{R}_-^{2n})$  and  $\sigma \in \mathbb{R}$  then*

$$\sum_{k=0}^\infty 2^{2k\sigma} ||[P_k, f]u||^2 \lesssim |||u|||_{\sigma-1}^2.$$

**Lemma 4.3.** *Let  $L$  be a tangential vector field with coefficients in  $C_0^\infty(\mathbb{R}_-^{2n})$ . Then*

$$\sum_{k=0}^\infty ||[P_k, L]u||^2 \leq C||u||^2.$$

The proof of Lemma 4.2 and Lemma 4.3 can be found in [3]. We remark that if we replace  $u \in \mathcal{S}(\mathbb{R}_-^{2n})$  by  $u \in C^\infty(\bar{D} \cap U)^k \cap D_{\bar{\partial}*}$ , then the two lemmas above still hold.

*Proof of Theorem 1.6* It suffices to prove the weaker version of (1.3) in which  $|| \cdot ||_\epsilon$  is replaced by  $||| \cdot |||_\epsilon$ . In fact,  $D_r$  can be expressed as a linear combination of  $\bar{L}_n$  and a suitable “totally real tangential” vector field that we denote by  $T$ . We have

$$\begin{cases} Q(u, u) \gtrsim ||\bar{L}_n u||^2, \\ |||u|||_\epsilon^2 \gtrsim ||Tu||_{\epsilon-1}^2. \end{cases}$$

It follows

$$\begin{aligned} \|u\|_\epsilon^2 &= \|D_r(u)\|_{\epsilon-1}^2 + \|u\|_\epsilon^2 \\ &\lesssim Q(u, u) + \|u\|_\epsilon^2, \end{aligned}$$

which proves the claim. By Lemma 4.1 and Theorem 3.2, we get for any  $u \in C^\infty(\bar{D} \cap V)^k \cap \text{Dom}(\bar{\partial}^*)$  with  $k \geq q+1$

$$\begin{aligned} \|u\|_\epsilon^2 &\lesssim \sum_{j=0}^q \|L_j u\|_{\epsilon-1}^2 + \sum_{j=q+1}^n \|\bar{L}_j u\|_{\epsilon-1}^2 + \|u_b\|_{\epsilon-1/2}^2 \\ &\lesssim Q(u, u) + \|u_b\|_{\epsilon-1/2}^2. \end{aligned}$$

Now, we estimate  $\|u_b\|_{\epsilon-1/2}^2$ . Let  $\chi_k \in C_c^\infty(-2^{-k}, 0]$  with  $0 \leq \chi_k \leq 1$  and  $\chi_k(0) = 1$ . We have the elementary inequality

$$|g(0)|^2 \leq \frac{2^k}{\eta} \int_{-2^{-k}}^0 |g(r)|^2 dr + 2^{-k} \eta \int_{-2^{-k}}^0 |g'(r)|^2 dr,$$

which holds for any  $g$  such that  $g(-2^{-k}) = 0$ . If we apply it for  $g(r) = \chi_k(r) P_k u(\cdot, r)$ , we get

$$\begin{aligned} \|u_b\|_{\epsilon-1/2}^2 &\cong \sum_{k=0}^{\infty} 2^{2k(\epsilon-1/2)} \|\chi_k(0) P_k u_b\|^2 \\ &\leq \eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^0 \|\chi_k P_k u(\cdot, r)\|^2 dr + \eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^0 \|D_r(\chi_k P_k u(\cdot, r))\|^2 dr \\ &= \underbrace{\eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^0 \|\chi_k P_k u(\cdot, r)\|^2 dr}_I + \underbrace{\eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^0 \|D_r(\chi_k P_k u(\cdot, r))\|^2 dr}_{II} \\ &\quad + \underbrace{\eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \int_{-2^{-k}}^0 \|\chi_k D_r(P_k u(\cdot, r))\|^2 dr}_{III}. \end{aligned}$$

Observe that  $\chi_k \leq 1$  and recall Theorem 3.2 that we apply for  $P_k u$  and  $\delta = 2^{-k}$ . Thus the first sums above can be estimated by

$$\begin{aligned} (I) &\leq \eta^{-1} \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^0 \|P_k u(\cdot, r)\|^2 dr \\ &\lesssim \eta^{-1} \sum_{k=0}^{\infty} Q(P_k u, P_k u). \end{aligned}$$

We note that  $Q(w, w)$  can be written as a finite sum of terms of the type

$$M_i = a_i T_i + b_i D_r + c_i,$$

where  $T_i$  are tangential vector fields. Hence

$$\begin{aligned} \sum_{k=0}^{\infty} Q(P_k u, P_k u) &\leq \sum_{k=0}^{\infty} \left( \|P_k \bar{\partial} u\|^2 + \|P_k \bar{\partial}^* u\|^2 \right) + \sum_i \sum_{k=0}^{\infty} \|[M_i, P_k]u\|^2 \\ &\lesssim Q(u, u) + \sum_i \sum_{k=0}^{\infty} \|[a_i T_i, P_k]u\|^2 + \sum_i \sum_{k=0}^{\infty} \|[b_i, P_k]D_r(u)\|^2 + \|u\|_{-1}^2 \\ &\lesssim Q(u, u) + \|u\|^2 + \|D_r(u)\|_{-1}^2, \end{aligned}$$

where the estimates on the commutator terms follow by Lemma 4.2 and Lemma 4.3. As it has already been remarked,  $D_r(u)$  can be expressed as a linear combination of  $\bar{L}_n u$  and  $Tu$  for some tangential vector field  $T$ . Then

$$\begin{aligned} \|D_r(u)\|_{-1}^2 &\lesssim \|\bar{L}_n u\|_{-1}^2 + \|Tu\|_{-1}^2 \\ &\lesssim \|\bar{L}_n u\|^2 + \|u\|^2 \\ &\lesssim Q(u, u) \end{aligned}$$

where the last line follows from Theorem 3.2.

We now estimate (II). Since  $D_r(\chi_k) \leq 2^k$ , we get

$$(II) \leq \eta \sum_{k=0}^{\infty} 2^{2k\epsilon} \int_{-2^{-k}}^0 \|P_k u(\cdot, r)\|^2 dr \leq \eta \sum_{k=0}^{\infty} 2^{2k\epsilon} \|P_k u\|^2 \cong \eta \|u\|_{\epsilon}^2.$$

As for the term (III), we have  $D_r P_k = P_k D_r$  and  $\chi_k \leq 1$ . Also  $D_r = a\bar{L}_n + bT$  as before. Thus

$$\begin{aligned} (III) &\leq \eta \sum_{k=0}^{\infty} 2^{2k(\epsilon-1)} \|P_k D_r(u)\| \cong \eta \|D_r(u)\|_{\epsilon-1} \\ &\lesssim \eta (\|\bar{L}_n u\|_{\epsilon-1}^2 + \|Tu\|_{\epsilon-1}^2) \\ &\lesssim \eta Q(u, u) + \eta \|u\|_{\epsilon}^2. \end{aligned}$$

Combining all our estimates of  $\|u_b\|_{\epsilon-1/2}$ , we obtain

$$\|u_b\|_{\epsilon-1/2} \lesssim \eta^{-1} Q(u, u) + \eta \|u\|_{\epsilon}.$$

Summarizing up, we have shown that

$$\|u\|_{\epsilon} \lesssim \eta^{-1} Q(u, u) + \eta \|u\|_{\epsilon}^2.$$

Choosing  $\eta > 0$  sufficiently small, we can move the term  $\eta \|u\|_{\epsilon}^2$  into the left-hand-side and get

$$\|u\|_{\epsilon}^2 \lesssim Q(u, u).$$

The proof is complete.

□

## 5. PROOF OF THEOREM 1.7

We note that  $r = 2x_n + h$  is a graphing function and denote by  $z \mapsto z^*$  the projection  $\bar{D} \rightarrow \partial D$  along the  $x_n$ -axis. We denote by  $\partial r^\perp(z)$ ,  $z \in V$  the bundle orthogonal to  $\partial r = \omega_n$  and note that  $\partial r^\perp|_{\partial D} = T^{1,0}\partial D$ . We have the evident equalities

$$(5.1) \quad \begin{cases} (r_{ij}(z))_{ij=1}^{n-1} = (r_{ij}(z^*))_{ij=1}^{n-1}, \\ \partial r^\perp(z) = \partial r^\perp(z^*). \end{cases}$$

Thus (5.1) relates  $L_r|_{T^c\partial D}$  on  $\partial D \cap V$  to  $L_r|_{\partial r^\perp}$  on the whole of  $\bar{D} \cap V$ ; in particular, (1.4) passes from  $\partial D \cap V$  to the whole of  $\bar{D} \cap V$ . Since  $\tilde{r}$  is obtained by adding to  $r$  terms which in turn satisfy (1.4) in the two respective cases, then one can prove that  $\tilde{D}$  is  $q$ -pseudoconvex (resp.  $q$ -pseudoconcave) in the sense of its “exhaustion” functions (though this is not clear for defining functions). We do not enter into these details and just show, in the beginning of the proof that (5.1), which concerns the behavior of  $r$  on  $\partial D$ , turns into a similar property of a weight  $\varphi$  in  $\bar{D}$ .

We choose a local basis  $\omega_1, \dots, \omega_n = \partial \tilde{r}$  of  $(1, 0)$ -forms and denote by  $L_1, \dots, L_n$  the dual basis of  $(1, 0)$ -vector fields; we may assume that  $L_j(z_o) = \partial_{z_j}$ . Thus, by an orthonormal change in the system  $\text{Span}\{L_1, \dots, L_{n-1}\}$ , we can assume that (1.4) is satisfied on  $\partial D$ . We now construct the weight  $\varphi$  which satisfies the assumptions of Theorem 1.6; for this we distinguish  $q > q_o$  from  $q < q_o$ .

The case  $q$ -pseudoconvex. We set for a suitable constant  $\lambda > 0$

$$(5.2) \quad \psi = -\log(-\tilde{r} + \delta) + \lambda|z|^2 + \sum_{j=q}^{n-1} \log(|z_j|^2 + \delta^{\frac{1}{m_j}}),$$

and define  $\varphi := c|\log \delta|^{-1}\psi$  where  $c$  is an irrelevant constant needed to get the bound 1 in (1.6) and (1.7). We set  $\psi^I = -\log(-\tilde{r} + \delta) + \lambda|z|^2$  and denote by  $\psi^{II}$  the remaining term in the right of (5.2); thus  $\psi = \psi^I + \psi^{II}$ . We have

$$(5.3) \quad \begin{aligned} \psi_{ij}^I &= (-\tilde{r} + \delta)^{-1} \tilde{r}_{ij} + \lambda \kappa_{ij} \\ &= (-\tilde{r} + \delta)^{-1} r_{ij} + \lambda \kappa_{ij} + (-\tilde{r} + \delta)^{-1} (\partial_{z_j} \partial_{\bar{z}_j} h_j) \kappa_{ij} + \mathcal{E} \quad \text{for } i, j \leq n-1, \end{aligned}$$

where  $\mathcal{E}$  is an error of type  $\mathcal{E} = O(|z|)(-\tilde{r} + \delta)^{-1} \sum_j (\partial_{z_j} \partial_{\bar{z}_j} h_j)$ . We also have

$$(5.4) \quad \psi_{nn}^I = (-\tilde{r} + \delta)^{-2}.$$

(where  $\kappa_{ij}$  continues to denote the Kronecker's symbol) and

$$(5.5) \quad \psi_{ij}^{II} = \left( \frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \right) \kappa_{ij}$$

When taking  $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$  of  $(-\tilde{r} + \delta)^{-1} r_{ij} + \lambda \kappa_{ij}$  from (5.3) and of  $(-r + \delta)^{-2}$  from (5.4) the result is  $\geq 0$ . This is true for  $(r_{ij}(z))_{ij}|_{\partial r^\perp(z)}$  on account of (5.1). But what is left is just

$$\lambda |\omega|^2 + (-\tilde{r} + \delta)^{-2} |\partial r|^2 + (-\tilde{r} + \delta)^{-1} 2 \operatorname{Re} \sum_{j=1}^n r_{nj} \partial r \otimes \bar{\omega}_j,$$

which is positive. We also discard all terms of type  $(\partial_{z_j} \partial_{\bar{z}_j} h_j) \kappa_{ij}$  and  $\delta^{\frac{1}{m_j}} \kappa_{ij}$  for  $i$  or  $j \leq k-1$  in addition to  $\mathcal{E}$  because they can be made positive by adding a small amount of terms for which  $i, j \geq k$  on account of the estimates (5.7) and (5.8) which follow. For the remaining terms  $(\partial_{z_j} \partial_{\bar{z}_j} h_j)$ , we note that we have  $(\partial_{z_j} \partial_{\bar{z}_j} h_j) \gtrsim |z_j|^{2m_j-2}$ . We end up with the estimate

$$(5.6) \quad \sum'_{|K|=k-1} \sum_{ij=1}^n \psi_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_o} \psi_{jj} |u|^2 \\ \geq \sum_{j=k}^{n-1} \left( (-\tilde{r} + \delta)^{-1} |z_j|^{2m_j-2} + \frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \right) \sum'_{|K|=k-1} |u_{jK}|^2 + (-r + \delta)^{-2} \sum'_{|K|=k-1} |u_{nK}|^2.$$

We now inspect the coefficients in the right of (5.6). First, let  $z \in S_\delta$ , that is,  $-r > \delta$ . Given a coefficient  $u_J$  of  $u$ , the index  $J$  contains for sure at least one  $j$  such that  $k \leq j \leq n-1$  and thus  $u_J = \operatorname{sign} \binom{J}{jK} u_{jK}$  for a suitable  $K$ . If, for this  $j$ ,  $|z_j|^2 \geq \delta^{\frac{1}{m_j}}$ , then

$$(5.7) \quad (-\tilde{r} + \delta)^{-1} |z_j|^{2m_j-2} \gtrsim \delta^{-\frac{1}{m_j}}.$$

On the contrary, if  $|z_j|^2 \leq \delta^{\frac{1}{m_j}}$ , then

$$(5.8) \quad \frac{\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \gtrsim \delta^{-\frac{1}{m_j}}.$$

In both cases, the terms in the left are  $\geq \delta^{-2\epsilon_k}$  since  $-\frac{1}{m_j} \leq -\frac{1}{m_k} = -2\epsilon_k$ . By combining (5.7) with (5.8), we get the second of (1.7) for  $\epsilon = \epsilon_k$ . On the other hand, for any  $j \leq q_o$ , we have  $\frac{r_j}{-r+\delta} = 0$  and therefore  $\psi_j = \lambda O(|z|)$  which is estimated by  $\bar{\partial} \partial(\lambda |z|^2)$ . On the other hand,  $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$  is always  $\geq 0$  all over  $\bar{D} \cap V$ . This proves the second inequality in (1.6).

Finally, a normalization by a factor  $c |\log \delta|^{-1}$  makes the weight bounded as required by the first of (1.6) and (1.7), at the expenses of passing from  $\delta^{-2\epsilon_k}$  to  $\frac{\delta^{-2\epsilon_k}}{|\log \delta|}$  in (1.7). Thus the weight  $\psi$  satisfies all the requirements of Theorem 1.1 for any  $\epsilon < \epsilon_k$  which implies subelliptic estimates of the corresponding order. Incidentally, we notice that when

$\epsilon_k = \frac{1}{2}$ , the term  $\psi^{II}$  is needless and we can take a different normalization by defining  $\varphi = -\log\left(\frac{-r+\delta}{2\delta}\right)$ ; thus we get an even  $\delta^{-1}$  on the right of (1.7). For  $\epsilon_k = \frac{1}{2}$ , a similar argument applies also to the case  $q$ -pseudoconcave which follows and we will not insist on it.

The case  $q$ -pseudoconcave. We now define

$$\psi = -\log(-\tilde{r} + \delta) - \lambda|z|^2 + \sum_{j=1}^{k+1} \log(-\log(|z_j|^2 + \delta^{\frac{1}{m_j}}))$$

where we point out the attention to the double log. Comparing with the case  $q$ -pseudoconvex, there is now an extra difficulty for the weight to satisfy (1.6) (whereas (1.7) remains substantially unchanged) because we do not have any longer  $\varphi_j = 0$  for  $j \leq q_o$ . We write  $\psi = \psi^I + \psi^{II}$  in the same way as in the previous case and will eventually define  $\varphi$  by a normalization  $\varphi = c|\log \delta|^{-1}\psi$ . We have the analogous of (5.3) and (5.5) with the suitable sign. We apply  $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$  to  $\psi^I + \psi^{II}$ . When taking  $\sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot$  we discard the contribution of  $(-\tilde{r} + \delta)^{-1}r_{ij} + \lambda\kappa_{ij}$  in addition to the normal term  $(-\tilde{r} + \delta)^{-2}$  because this contribution is positive as before. We discard the error term  $\mathcal{E}$  because it can be made positive by the aid of a small amount of the remainder. This argument is the same as for the case  $q$ -pseudoconvex. What we are left with is

$$(5.9) \quad \sum_{ij} \cdot - \sum_{j=1}^{q_o} \cdot \geq \sum_{j=1}^{k+1} \left( (-r + \delta)^{-1}|z_j|^{2m_j-2} + \frac{-\delta^{\frac{1}{m_j}}}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \frac{1}{|\log(|z_j|^2 + \delta^{\frac{1}{m_j}})|} \right. \\ \left. + \frac{|z_j|^2}{(|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \frac{1}{|\log(|z_j|^2 + \delta^{\frac{1}{m_j}})|^2} \right) (|u|^2 - \sum'_{|K|=k-1} |u_{jK}|^2).$$

We write the coefficient in the right of (5.9) as  $(A_j + B_j + C_j)$ . The two first terms serve to get (1.7), the third for (1.6). (This latter was discarded as  $\geq 0$  in the case  $q$ -pseudoconvex; here it is essential because  $\varphi_j \neq 0$  for  $j \leq q_o$ ). Reasoning as in the first half of the proof we get, for any  $j \leq k+1$

$$(5.10) \quad A_j + B_j \underset{\sim}{\geq} \delta^{-\frac{1}{m_j}} \geq \delta^{-2\epsilon_k} \quad \text{on } S_\delta \cap V,$$

because  $-\frac{1}{m_j} \leq -\frac{1}{m_{k+1}} = -2\epsilon_k$  for any  $j \leq k+1$ , along with

$$(5.11) \quad A_j + B_j \geq 0 \quad \text{on } \bar{D} \cap V.$$

We make the crucial remark for the case of concavity. If the degree of  $u$  is  $k$ , then

$$(5.12) \quad \sum_{j=1}^{k+1} \left( |u|^2 - \sum'_{|K|=k-1} |u_{jK}|^2 \right) \geq |u|^2.$$



From (5.10) and (5.12) we get (1.7). We now need to prove that on  $\bar{D} \cap V$  and for a suitable  $\epsilon$  we have

$$\begin{aligned}
 (5.13) \quad & \sum_{j=1}^{k+1} \frac{1}{\log^2 \cdot (|z_j|^2 + \delta^{\frac{1}{m_j}})^2} \left( |u|^2 - \sum'_{|K|=k-1} |u_{jK}|^2 \right) \\
 & \geq \epsilon \sum_{j=1}^{k+1} \frac{|z_j|^2}{\log^2 \cdot (|z_j|^2 + \delta^{\frac{1}{m_j}})^2} |u|^2 \\
 & = \epsilon \sum_{j=1}^{k+1} |\varphi_j|^2 |u|^2.
 \end{aligned}$$

This would conclude the proof of (1.6). The last sum  $\sum_{j=1}^{k+1} \cdot$  can be replaced by  $\sum_{j=1}^{q_o} \cdot$  since  $\varphi_j = 0$  for  $j = k+2, \dots, q_o$ . Also, remember here that  $\psi_j^I = 0$  for any  $j$  and  $\psi_j^{II} = 0$  for any  $j \geq k+2$ ; this justifies the last equality in (5.13) which is true. However, the first inequality is wrong. To make it true, we need a small perturbation of  $\psi$ . We take a vector  $v$  in the unit sphere  $S^k$  outside the first quadrant, set  $\psi^{IIv} := \sum_{j=1}^{k+1} \log(-\log(|z_j|^2 + \delta^{\frac{1}{m_j}}) v_j)$ , leave  $\psi^I$  unchanged and define a new  $\psi$  by

$$\psi := \psi^I + \frac{1}{2}(\psi^{II} + \psi^{IIv}).$$

Inequalities (5.10) and (5.11) are stable under perturbation and thus will remain true for this new  $\psi$ . As for the first of (5.13), we consider the vector field

$$w(z) := \left( \frac{|z_j|}{\log^2 \cdot (|z_j|^2 + \delta^{\frac{1}{m_j}})} \right)_{j=1, \dots, k+1}.$$

We also define

$$\mu(z) = \frac{w(z)}{|w(z)|}, \quad \nu(z) = (\nu_j v_j)_{j=1, \dots, k+1};$$

thus  $|\mu| = 1$  and  $|\nu| \leq 1$ . Finally, we set

$$u = \frac{(|u_{jK}|)_{j=1, \dots, k+1}}{\sum_j |u_{jK}|^2}.$$

It suffices to prove that

$$\frac{1}{2} (\langle \mu, u \rangle^2 + \langle \nu, u \rangle^2) \leq 1 - \epsilon.$$

Now, we begin by noticing that

$$(5.14) \quad \begin{cases} \langle \mu, u \rangle \leq 1, \\ \langle \nu, u \rangle \leq 1, \end{cases}$$

by Cauchy-Schwartz inequality. Also, if the first of (5.14) happens to be equality, that is,  $\mu$  is parallel to  $u$ , then

$$\begin{aligned} \langle \nu, u \rangle &= \sum_j v_j \mu_j u_j \\ &= \sum_j v_j \mu_j^2. \end{aligned}$$

But for this to be 1 we need both  $\sum_j \mu_j^4 = 1$  and  $v$  parallel to  $(\mu_j^2)_{j=1, \dots, k+1}$ . If the first occurs then, since  $\sum_j \mu_j^2 = 1$ , we have  $(\mu_j^2) = (\mu_j)$  (and both coincide with a cartesian vector): thus  $(\mu_j^2)$  is not parallel to  $v$ . In conclusion if the first of (5.14) is equality, the second is not. Therefore, the function  $(u, \mu) \mapsto \frac{1}{2}(\langle u, \mu \rangle^2 + \langle u, \nu \rangle^2)$  has a minimum  $< 1$ , say  $1 - \epsilon$ , for  $u \in S^k$  (and for  $\nu = (\mu_j v_j)$ ).

□

## 6. PROOF OF THEOREM 1.11

**Lemma 6.1.** *We have*

$$(6.1) \quad \int_0^\delta \dots \int_0^\delta \frac{dx_1 dy_1 \dots dx_p dy_p}{(t \sum_{j=1}^p |t^{-\varepsilon} z_j|^{2m_j} + 1)^s} \cong t^{-\sum_1^p \frac{1}{m_j} + 2p\varepsilon},$$

provided that  $s > \frac{1}{m_1} + \dots + \frac{1}{m_p} + 1$ .

*Proof.* We can assume that  $m_1 \leq m_2 \leq \dots \leq m_p$ . Put  $a(t) = t \sum_{j=2}^p |t^{-\varepsilon} z_j|^{2m_j} + 1$ . First, we perform integration

$$M(z_2, \dots, z_p) = \int_0^\delta \int_0^\delta \frac{dx_1 dy_1}{(t |t^{-\varepsilon} z_1|^{2m_1} + a(t))^s}.$$

We also make a change of variables  $z'_1 = t^{\frac{1}{2m_1} - \varepsilon} a(t)^{-\frac{1}{2m_1}} z_1$  and get

$$M(z_2, \dots, z_p) = a(t)^{-s + \frac{1}{m_1}} t^{-\frac{1}{m_1} + 2\varepsilon} \int_0^{t^{\frac{1}{2m_1} - \varepsilon} a(t)^{-\frac{1}{2m_1}} \delta} \int_0^{t^{\frac{1}{2m_1} - \varepsilon} a(t)^{-\frac{1}{2m_1}} \delta} \frac{dx'_1 dy'_1}{(|z'_1|^{2m_1} + 1)^s}.$$

Since

$$t^{\frac{1}{2m_1} - \varepsilon} a(t)^{-\frac{1}{2m_1}} \delta = \left( \frac{t^{1-2\varepsilon m_1} \delta^{2m_1}}{\sum_{j=2}^p t^{1-2\varepsilon m_j} |z_j|^{2m_j} + 1} \right)^{\frac{1}{2m_1}} \geq C > 0,$$

then

$$M(z_2, \dots, z_p) \cong a(t)^{-q + \frac{1}{m_1}} t^{-\frac{1}{m_1} + 2\varepsilon}.$$

In conclusion, the left hand side of (6.1) is equivalent to

$$t^{-\frac{1}{m_1}+2\varepsilon} \int_0^\delta \dots \int_0^\delta \frac{dx_2 dy_2 \dots dx_p dy_p}{(t \sum_{j=2}^p |t^{-\varepsilon} z_j|^{2m_j} + 1)^{q-\frac{1}{m_1}}}.$$

Repetition of this argument for  $z_2, \dots, z_p$  yields the proof of the lemma.  $\square$

*Proof of Theorem 1.11 (i)* Let

$$\begin{cases} \omega_j = dz_j - r_{\bar{z}_j} dz_n, \\ \omega_n = \partial r \end{cases}$$

be a basis of  $(1, 0)$  forms. We note that for a  $k$ -form  $u$  we have  $u \in D_{\bar{\partial}^*}$  if and only if its coefficients satisfy  $u_{nK}|_{\partial D} \equiv 0$  for any  $|K| = k - 1$ . Let  $L_j$  be the dual basis of  $(1, 0)$  vector fields; these are a perturbation of  $\partial_{z_j} - r_{z_j} \partial_{z_n}$   $j = 1, \dots, n - 1$  and  $\sum_{j=1}^n r_{z_j} \partial_{z_j}$ . We have

$$\begin{aligned} & \bullet (\omega_{iK}, \omega_{jK}) = \kappa_{ij} + r_{z_i} r_{\bar{z}_j} \text{ for any } i, j \leq n - 1, \\ & \bullet (\omega_{jK}, \omega_{nK}) = 0 \text{ for any } j \leq n - 1, \\ & \bullet (\omega_I, \omega_J) = 0 \text{ if } |I \cap J| \leq k - 2, \\ & \bullet (\bar{\partial}^* u)_K = \sum_{j=1}^n \sum_{J: |J \cap jK|=k} L_j(u_J) + \sum_{j=1}^{n-1} \sum_{J: |J \cap jK|=k-1} L_j(u_J) \left( O(r_{z_j}) + \sum_{i \in J} O(r_{z_i}) \right) \\ & \quad + \text{error}, \end{aligned}$$

where “error” denotes a term where no derivatives of  $u$  occur. We will deal with the form

$$u_t = U_t \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_k,$$

where  $U_t$  is a functions which will be specified later. We have for this form

$$\begin{cases} \bar{\partial} u \simeq \sum_{j=k+1}^n \bar{L}_j(U_t) \bar{\omega}_j \wedge \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_k + \text{error}, \\ \bar{\partial}^* u = \sum_{j=1}^k L_j(U_t) \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_j \wedge \dots \wedge \bar{\omega}_k + \sum_{j=1}^{n-1} \sum_{H \cap \{1, \dots, k, j\} \neq \emptyset} L_j(U_t) \left( O(r_{z_j}) + \sum_{i \leq k} O(r_{z_i}) \right) \bar{\omega}_H \\ \quad + \text{error}. \end{cases}$$

In particular

$$\begin{aligned} (6.2) \quad & \|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2 \lesssim \sum_{j=k+1}^n \|\bar{L}_j U_t\|^2 + \sum_{j=1}^k \|L_j U_t\|^2 \\ & + \sum_{i=k+1, \dots, n-1} \left\| \left( O(|r_{z_i}|) + \sum_{i \leq k} O(|r_{z_j}|) \right) L_i U_t \right\|^2 + \|U_t\|^2. \end{aligned}$$

We now set  $U_t = f_t(z_n)\Phi_t(z)$  where

$$\begin{cases} f_t(z', z_n) = (z_n - Q(z') - \frac{1}{t})^{-p} \\ \Phi_t(z) = (\prod_{j=1}^{n-1} \varphi(t^{\epsilon_k} x_j) \varphi(t^{\epsilon_k} y_j)) \lambda(x_n) \varphi(y_n). \end{cases}$$

Here  $\varphi \in C_0^\infty(\mathbb{R})$  satisfies

$$\varphi(x) = \begin{cases} 1 & x \leq \delta \\ 0 & x \geq 2\delta, \end{cases}$$

where  $\delta$  is a small parameter, and  $\lambda \in C_0^\infty(\mathbb{R})$  will be chosen later. Since

$$\begin{cases} L_j(f_t) = 0 & \text{for any } j \leq q_o, \\ \bar{L}_j(f_t) = 0 & \text{for } q_o + 1 \leq j \leq n-1, \\ \partial_{z_j}(f_t) = \partial_{\bar{z}_j}(f_t) = 0 & \text{for } j = q_o + 1, \dots, n-1, \end{cases}$$

we can restrict the first sum in (6.2) to  $j = n$  and the second to  $j = q_o + 1, \dots, k$ ; thus we get

(6.3)

$$\begin{aligned} Q(u_t, u_t) &\lesssim \sum_{ij=1}^{q_o} \|r_{z_i} \partial_{\bar{z}_j}(f_t) \Phi_t\|^2 + \sum_{j=q_o+1}^k \|r_{z_j} \partial_{z_n}(f_t) \Phi_t\|^2 + \sum_{\substack{i=k+1, \dots, n-1 \\ j=1, \dots, k, i}} \|r_{z_j} \|r_{z_i} \partial_{z_n}(f_t) \Phi_t\|^2 \\ &+ \sum_{ij=1}^{n-1} \|O^2(|r_{z_i}|) \partial_{z_j} f_t \Phi_t\|^2 + \sum_{j=1}^n \|f_t \partial_{z_j} \Phi_t\|^2 + \|U_t\|^2. \end{aligned}$$

To estimate the first three sums in (6.3) we need to evaluate  $r_{z_i}$  for  $i = 1, \dots, q_o$ , next  $r_{z_j}$  for  $j = q_o + 1, \dots, k$  and finally  $r_{\bar{z}_i} r_{z_j} r_{z_i}$  for  $i = k + 1, \dots, n-1$ ,  $j = 1, \dots, k$ . We perform the change of variables

$$\begin{cases} \tilde{z}_j = t^{\epsilon_k} z_j, & j \leq n-1, \\ \tilde{z}_n = t z_n. \end{cases}$$

For  $|\tilde{z}| \leq 1$  we have for the first terms

$$\begin{aligned} (6.4) \quad |r_{z_j}(z)| &= |z_j|^{4m_j-2} \\ &= t^{\epsilon_k(4m_j-2)} \leq t^{-2+2\epsilon_k}, \quad j = q_o + 1, \dots, k, \end{aligned}$$

where the last inequality follows from  $m_j \geq m_k$ . For the second terms we have

$$\begin{aligned} (6.5) \quad |r_{z_i}(z')|^2 &\leq |z'|^{4m-2} \\ &= t^{\epsilon_k(4m-2)} \leq t^{-2+2\epsilon_k}, \quad i = 1, \dots, q_o, \end{aligned}$$

where the last estimate follows from  $m \geq m_{q_0+1} \geq m_k$ . For the third terms we extend the definition of  $m_j$  to  $j \leq q_0$  by putting  $m_j = m$ . We have, for  $i \geq k+1$ ,  $j \leq k$  or  $j = i$

$$(6.6) \quad \begin{aligned} |r_{z_j}(z)|^2 |r_{z_i}(z)|^2 &\leq t^{-\epsilon_k(4m_i+4m_j-4)} \\ &\leq t^{-2-\epsilon_k(4m_j-4)} \leq t^{-2}, \end{aligned}$$

where the second inequality follows from  $m_i \leq m_k$ . If we pass to estimate the terms in the second sum of (6.3), we then have

$$\begin{aligned} \|r_{z_j} \frac{\partial f_t}{\partial z_n} \Phi_t\|^2 &\cong \int |r_{z_j}|^2 |z_n - Q(z') - \frac{1}{t}|^{-2p-2} \Phi_t^2(z) dx_1 dy_1 \dots dx_n dy_n \\ &\lesssim \int \frac{|z_j|^{4m_j-2}}{\left((\frac{1}{t} + Q(z') - x_n)^2 + y_n^2\right)^{p+1}} \Phi_t^2(z) dx_1 dy_1 \dots dx_n dy_n \\ &\lesssim t^{2p-2+2\epsilon_k-2(n-1)\epsilon_k} I_t, \end{aligned}$$

where

$$I_t = \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 \lambda(t^{-1} \tilde{x}_n)^2 \varphi(t^{-1} \tilde{y}_n)^2}{\left((1 + tQ(t^{-\epsilon_k} \tilde{z}') - \tilde{x}_n)^2 + \tilde{y}_n^2\right)^{p+1}} d\tilde{x}_1 d\tilde{y}_1 \dots d\tilde{x}_n d\tilde{y}_n.$$

We now perform integration in  $\tilde{y}_n$  from  $-\infty$  to  $+\infty$  and get

$$I_t \lesssim \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 \lambda(t^{-1} \tilde{x}_n)^2}{(1 + tQ(t^{-\epsilon_k} \tilde{z}') - \tilde{x}_n)^{2p+1}} d\tilde{x}_1 d\tilde{y}_1 \dots d\tilde{x}_{n-1} d\tilde{y}_{n-1} d\tilde{x}_n.$$

Next, we integrate in  $\tilde{x}_n$  from

$$-\infty \text{ to } \left(tQ(t^{-\epsilon_k} \tilde{z}') - t \sum_{j=q_0+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2\right)/2,$$

and get

$$\begin{aligned} I_t &\lesssim \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2}{(tQ(t^{-\epsilon_k} \tilde{z}') + t \sum_{j=q_0+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 2)^{2p}} d\tilde{x}_1 d\tilde{y}_1 \dots d\tilde{x}_{n-1} d\tilde{y}_{n-1} \\ &\lesssim \int_0^{2\delta} \dots \int_0^{2\delta} \frac{d\tilde{x}_{k+1} d\tilde{y}_{k+1} \dots d\tilde{x}_{n-1} d\tilde{y}_{n-1}}{(t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} z_j|^{2m_j} + 1)^{2p}} \\ &\lesssim t^{-\sum_{j=k+1}^{n-1} \frac{1}{m_j} + 2(n-k-1)\epsilon_k} \end{aligned}$$

where the last inequality follow by Lemma 6.1.

In conclusion we have obtained

$$(6.7) \quad \|r_{z_j} \frac{\partial f_t}{\partial z_n} \Phi_t\|^2 \lesssim t^{2p-2+2\epsilon_k-2k\epsilon_k-\sum_{j=k+1}^{n-1} \frac{1}{m_j}}.$$

The same integration combined with (6.6) yields the same estimate as (6.7) also for the terms  $|||r_{z_j}||r_{z_i}|\partial_{z_n}(f_t)\Phi_t||^2$  for  $i \geq k+1$  and  $j \leq k$ . As for the terms in the first sum in (6.3) with  $i, j = 1, \dots, q_0$ , we have

$$\begin{aligned} \|r_{z_j} \frac{\partial f_t}{\partial \bar{z}_j}\|^2 &\cong \int |r_{z_j}|^4 |z_n - Q(z') - \frac{1}{t}|^{-2p-2} \Phi_t^2(z) dx_1 dy_1 \dots dx_n dy_n \\ &\lesssim \int \frac{|z'|^{4m_k-2}}{\left(\left(\frac{1}{t} + Q(z') - x_n\right)^2 + y_n^2\right)^{p+1}} \Phi_t^2(z) dx_1 dy_1 \dots dx_n dy_n \\ &\lesssim t^{2p-2+2\epsilon_k-2k\epsilon_k-\sum_{j=k+1}^{n-1} \frac{1}{m_j}} \end{aligned}$$

where the last inequality follows by the same technique as above.

By the same argument all the sums  $\sum_{ij=1}^{n-1} ||O^2(|r_{z_i}|)\partial_{z_j} f_t \Phi_t||^2$ , the terms  $\|f_t \frac{\partial \Phi_t}{\partial z_j}\|^2$   $j = 1, \dots, n$  and  $\|U_t\|^2$  have the same estimate in terms of  $t$ . Combining all these estimates, we get the basic estimate from above for  $Q(u_t, u_t)$

$$(6.8) \quad Q(u_t, u_t) \lesssim t^{2p-2+2\epsilon_k-2k\epsilon_k-\sum_{j=k+1}^{n-1} \frac{1}{m_j}}.$$

To calculate  $|||u_t|||_\epsilon$  we use the boundary coordinates  $(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r)$  and dual coordinates  $(\xi, r) = (\xi_1, \dots, \xi_{2n-1}, r)$ . We have

$$\begin{aligned} |||u_t|||_\epsilon &= |||U_t|||_\epsilon^2 + \sum_{j=1}^k |||r_i U_t|||_\epsilon^2 \geq |||U_t|||_\epsilon^2 \\ &\geq \int (1 + |\xi|^2)^\epsilon |\hat{U}_t(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r)|^2 d\xi dr \\ &\geq \int |\xi_{2n-1}^{2\epsilon}| \left| \int \frac{\varphi(y_n) \lambda(x_n) e^{-i\xi_{2n-1} y_n} dy_n}{\left((x_n - Q(z') - 1/t + iy_n)^p\right)} \right|^2 \\ &\quad \cdot \left( \prod_{j=1}^k \varphi(t^{\epsilon_k} x_j) \varphi(t^{\epsilon_k} y_j) \right)^2 \left( \prod_{j=k+1}^{n-1} \varphi(x_j) \varphi(y_j) \right)^2 dx' dx'' dy' dy'' d\xi_{2n-1} dr, \end{aligned}$$

where we use Plancherel's theorem on  $\xi_1, \dots, \xi_{2n-2}$  in the second line. Similarly as before, we use transformations

$$\begin{cases} \tilde{x}_j = t^{\epsilon_k} x_j, & \tilde{y}_j = t^{\epsilon_k} y_j, \quad j = 1, \dots, n-1, \\ \tilde{y}_n = t y_n, & \tilde{\xi}_{2n-1} = 1/t \xi_{2n-1}, \quad \tilde{r} = tr, \end{cases}$$

and obtain

$$|||u_t|||_\epsilon^2 \geq t^{2p-2+2\epsilon-2(n-1)\epsilon_k} J_t,$$

where

$$J_t = \int |\tilde{\xi}_{2n-1}|^{2\epsilon} \left| \int \frac{\varphi(t^{-1}\tilde{y}_n)\lambda(x_n(t^{-\epsilon_k}\tilde{x}_1, \dots, t^{-1}\tilde{r}))e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n}d\tilde{y}_n}{(-g+i\tilde{y}_n)^p} \right|^2 \\ \cdot (\Pi_{j=1}^{n-1}\varphi(\tilde{x}_j)\varphi(\tilde{y}_j))^2 d\tilde{x}'d\tilde{x}''d\tilde{y}'d\tilde{y}''d\tilde{\xi}_{2n-1}d\tilde{r}.$$

Here

$$g = -\left(\frac{\tilde{r} - tQ(t^{-\epsilon_k}\tilde{z}') - t\sum_{j=q_0+1}^{n-1}|h_j(t^{-\epsilon_k}\tilde{z}_j)|^2}{2} - 1\right).$$

Since  $tQ(t^{-\epsilon_k}\tilde{z}') + t\sum_{j=q_0+1}^k|h_j(t^{-\epsilon_k}\tilde{z}_j)|^2 \lesssim \sum_{j=1}^{q_0}|\tilde{z}_j|^{2m} + \sum_{j=q_0+1}^k|\tilde{z}_j|^{2m_j}$ , then if the support of  $\varphi$  is small enough we can assume

$$tQ(t^{-\epsilon_k}t) + t\sum_{j=n-k+q_0}^{n-1}|h_j(t^{-\epsilon_k}\tilde{z}_j)|^2 \leq 1.$$

This implies  $0 < g \leq \frac{-\tilde{r} + t\sum_{j=k+1}^{n-1}|h_j(t^{-\epsilon_k}\tilde{z}_j)|^2 + 3}{2}$ . Using a further substitution

$$y'_n = g\tilde{y}_n, \quad \xi'_{2n-1} = \frac{1}{g}\tilde{\xi}_{2n-1},$$

we get

$$J_t = \int \frac{|\xi'_{2n-1}|^{2\epsilon}}{g^{2p+1-2\epsilon}} \left| \int \frac{\varphi(\frac{y'_n}{tg})\lambda(x_n)e^{-i\xi'_{2n-1}y'_n}dy'_n}{(-1+i\tilde{y}_n)^p} \right|^2 \\ \cdot (\Pi_{j=1}^{n-1}\varphi(\tilde{x}_j)\varphi(\tilde{y}_j))^2 d\tilde{x}'d\tilde{x}''d\tilde{y}'d\tilde{y}''d\tilde{\xi}_{2n-1}d\tilde{r}, \\ = J_1 + J_2$$

where  $J_1$  is the integration from  $-\infty$  to  $-tK$ ,  $J_2$  from  $-tK$  to 0 and where  $K$  is suitably chosen. Note that  $J_1 \geq 0$ . Now, we consider  $J_2$ .

For  $\tilde{r} \in [-tK, 0]$ , we see that

$$|x_n| = \left| \frac{\tilde{r}/t + Q(t^{-\epsilon_k}\tilde{z}') - \sum_{j=q_0+1}^{n-1}|h_j(t^{-\epsilon_k}\tilde{z}_j)|^2}{2} \right| \leq C.$$

We may choose  $\lambda \in C_0(\mathbb{R})$  such that  $\lambda(x) = 1$  for  $|x| \leq C$ . Then

$$\int |\xi'_{2n-1}|^{2\epsilon} \left| \int \frac{\varphi(\frac{y'_n}{tg})\lambda(x_n)e^{-i\xi'_{2n-1}y'_n}dy'_n}{(-1+i\tilde{y}_n)^p} \right|^2 d\xi'_{2n-1} \geq \text{const} > 0.$$

It follows

$$\begin{aligned}
J_2 &\gtrsim \int \int_{\tilde{r}=-tK}^0 \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 d\tilde{x}' d\tilde{x}'' d\tilde{y}' d\tilde{y}''}{\left(-\tilde{r} + t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3\right)^{2p+1-2\epsilon}} d\tilde{r} \\
&\gtrsim \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 d\tilde{x}' d\tilde{x}'' d\tilde{y}' d\tilde{y}''}{\left(t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3\right)^{2p-2\epsilon}} d\tilde{r} \\
&\quad - \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 d\tilde{x}' d\tilde{x}'' d\tilde{y}' d\tilde{y}''}{\left(tK + t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3\right)^{2p-2\epsilon}} d\tilde{r} \\
&\gtrsim \int \frac{(\Pi_{j=1}^{n-1} \varphi(\tilde{x}_j) \varphi(\tilde{y}_j))^2 d\tilde{x}' d\tilde{x}'' d\tilde{y}' d\tilde{y}''}{\left(t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3\right)^{2p-2\epsilon}} d\tilde{r}.
\end{aligned}$$

The last inequality follows from the fact that we can choose  $K$  and  $t$  such that

$$tK + t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3 \geq 2 \left( t \sum_{j=k+1}^{n-1} |h_j(t^{-\epsilon_k} \tilde{z}_j)|^2 + 3 \right).$$

Then

$$J_t \gtrsim \int_0^\delta \cdots \int_0^\delta \frac{d\tilde{x}_{k+1} d\tilde{y}_{k+1} \cdots d\tilde{x}_{n-1} d\tilde{y}_{n-1}}{(t \sum_{j=k+1}^{n-1} |t^{-\epsilon_k} \tilde{z}_j|^{2m_j} + 1)^{2p-2\epsilon}} \cong t^{-\sum_{j=k+1}^{n-1} \frac{1}{m_j} + 2(n-k-1)\epsilon_k},$$

where the last inequality follows by Lemma 6.1. So we have

$$(6.9) \quad |||u_t|||_\epsilon^2 \gtrsim t^{2p-2+2\epsilon-2k\epsilon_k-\sum_{j=k+1}^{n-1} \frac{1}{m_j}}.$$

Since subelliptic estimates hold with order  $\epsilon$  for any  $k$ -form ( $q_0 + 1 \leq k \leq n-1$ ), then

$$(6.10) \quad |||u_t|||_\epsilon^2 \lesssim Q(u_t, u_t).$$

Combining (6.8) (6.9) and (6.10), we get  $\epsilon \leq \epsilon_k$

The proof of Theorem 1.11 (i) is complete.  $\square$

*Proof of Theorem 1.11 (ii)* We proceed in similar way as in the proof of Theorem 1.11 (i) and choose the coefficient of our form by setting

$$\begin{cases} f_t(z', z_n) = (z_n - \sum_{j=1}^{q_0} |h_j(z_j)|^2 - 1/t)^{-p} \\ \Phi_t(z) = (\Pi_{j=1}^{n-1} \varphi(t^{\epsilon_k} x_j) \varphi(t^{\epsilon_k} y_j)) \lambda(x_n) \varphi(y_n) \end{cases}$$



Then

$$\begin{aligned} Q(u_t, u_t) &\lesssim \sum_{j=1}^{q_0} \|r_{z_j} \frac{\partial f_t}{\partial \bar{z}_j} \Phi_t\|^2 + \sum_{j=k+1}^{q_0} \left\| \frac{\partial f_t}{\partial \bar{z}_j} \Phi_t \right\|^2 \\ &\quad + \sum_{\substack{i=k+1, \dots, n-1 \\ j=1, \dots, k, i}} \|r_{z_j} \|r_{z_i} |\partial_{z_n}(f_t) \Phi_t|\|^2 + \sum_{j=1}^n \|f_t \frac{\partial \Phi_t}{\partial z_j}\|^2 + \|U_t\|^2. \end{aligned}$$

We can show that  $Q(u_t, u_t) \lesssim t^{2p-2+2\epsilon_k-2(n-1)\epsilon_k} I_t$  where

$$I_t = \int_0^\delta \dots \int_0^\delta \frac{dx_1 dy_1 \dots dx_k dy_k}{\left( t \sum_{j=1}^k |t^{-\epsilon_k} z_j|^{2m_j} + 1 \right)^{2p-2}}.$$

Owing to Lemma 6.1 we have  $I_t \lesssim t^{-\sum_{j=1}^k \frac{1}{m_j} + 2k\epsilon_k}$  which yields

$$Q(u_t, u_t) \lesssim t^{2p-2+2\epsilon_k-2(n-k-1)\epsilon_k - \sum_{j=1}^k \frac{1}{m_j}}.$$

Similarly, we have

$$\|u_t\|_\epsilon^2 \gtrsim t^{2p-2+2\epsilon_k-2(n-k-1)\epsilon_k - \sum_{j=1}^k \frac{1}{m_j}},$$

which yields the conclusion of the proof of Theorem 1.11 (ii). □

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